Chapter 3
RETA Model for Teaching Mathematics:
From the United Kingdom to Turkey

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ABSTRACT

For a long time, research into students' geometry performance has been regarded as an essential research topic in mathematics. By outlining an innovative mathematics teaching model and offering sample lesson practices from geometry on how to implement the model in the Turkish setting, the purpose of this chapter is to give an international perspective on reform-based practice in mathematics teaching and learning. The chapter focuses on a model for teaching geometry classes that incorporates realistic, exploratory, technology-enhanced, and active (RETA) principles, as well as its implementations in Turkish middle schools. It presents the different approaches of teaching geometry, common geometry classroom practices in Turkey, the previous models leading to the RETA model, and finally, a review of the RETA model's principles together with their benefits and drawbacks followed by a discussion.

INTRODUCTION

This chapter provides an international perspective on a reform-based practice in mathematics teaching and learning by describing a novel mathematics teaching model, and presenting sample lesson practices on how to use the model in the Turkish context. Specifically, this chapter introduces a model to teach geometry lessons with realistic, exploratory, technology-enhanced and active (RETA) principles, and its practical applications in middle schools in Turkey. First, both the alternative models of teaching geometry and the typical geometry teaching approach in Turkey are discussed. Then, how these existing models led to the RETA model is described, followed by a discussion of the principles of the RETA model along with pros and cons. For the pros and cons of each principle of the model, practical applications of the principles are presented through sample lesson plans. These sample RETA-based lesson plans are on

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orthogonal and isometric drawings of polycubes which were developed, implemented, and evaluated as a part of a design-based research project with four cycles. The lessons were intended to help middle school students learn about three-dimensional (3D) shapes and in particular orthogonal and isometric drawings of polycubes. Last, the chapter ends with a discussion of the solutions, recommendations, and future research directions.

The 3D geometry thinking is "the conception of thoughts and ideas about 3D geometry concepts by amalgamating various types of reasoning;" and reasoning in this concept refers to "a set of processes and abilities that act as a feasible tool in problem-solving and enable us to go beyond the information given" (Pittalis & Christou, 2010, p.192). Orthogonal and isometric drawings have a variety of names in the literature; for example, orthogonal drawings can be referred to as orthogonal projections (Jones et al., 2012), orthographic projections/drawings (Moyer-Packenham & Bolyard, 2002), plan/top view, and elevations/side views (Yeo et al., 2005), and isometric drawings are also known as isometric projections (Gambari et al., 2015) and perspective drawings (Oldknow & Tetlow, 2008), as well as sometimes very vaguely as a building or a representation/picture of a building (Ben-Haim et al., 1985). The differing titles for these 2D representations were deemed synonyms in this chapter, and it is opted to use orthogonal and isometric drawings, which are both among the earliest names for these sorts of representations in the literature (Cooper & Sweller, 1989). In a similar vein, polycubes are three-dimensional shapes that are constructed from unit cubes. Polycubes are also referred to as polycubical shapes/objects (Cooper & Sweller, 1989), (solid) cube constructions (Ben-Haim et al., 1985) and a solid or an object constructed from unit-sized cubes (Pittalis & Christou, 2010).

BACKGROUND

Models to Teach Geometry

Recently, there have been various attempts to develop frameworks to teach mathematics at the middle school level, as Usiskin (1987) did in the 1980s. Particularly, the focus of research on 2D representations of 3D shapes has been on developing frameworks that characterize and analyze 3D geometry thinking. Of this research, some helped to improve the RETA model which is introduced in this chapter.

For example, Yeh and Nason (2004) suggested and tested a framework for using technology to teach 3D geometry. They claimed that three separate components make up 3D geometry: communication, objects, and spatial thinking. In a technological context, communication refers to (a) spoken and written language to describe 3D geometric entities and (b) non-verbal 2D representation of objects. Taking these three components into account, they designed VRMath, a software program that displayed realistic representations of 3D geometry problems in a range of colours, along with a link to a discussion forum. The authors state that two primary school students in grades 6 and 7 found the tool effective in promoting the construction of knowledge about 3D geometry ideas and procedures.

Recently, Goodall and colleagues (2017) proposed a framework for teaching mathematics that many students in the United Kingdom can use at home or school (not specified in but including 3D geometry). In contrast to the tedious, isolated, rote, elitist, and depersonalized (TIRED) mathematics discovered by Nardi and Steward (2013), they believe mathematics teaching should be accessible, linked, inclusive, valued, and empowering (ALIVE). To comprehend ALIVE, we must first comprehend the TIRED framework. In a research study conducted by Nardi and Steward (2013) with 70 13- and 14-year-old pupils in the United Kingdom, they discovered that mathematics instruction can be experienced by students as TIRED:

- Tedious: Math was tedious for the majority of the pupils, who saw it as a dull and unimportant subject with no real-world applications. Furthermore, interviewed pupils stated that mastering mathematics provides little opportunities for practical life.
- Isolated: Students thought of mathematics as a solitary topic in which students had to work alone to solve a mathematics problem.
- Rote: Many children saw mathematics as a series of rules to follow, therefore there were indisputable and distinct answers to math problems for them.
- Elitist: Pupils saw mathematics as a difficult subject and came to believe that only very bright or gifted students could achieve in it.
- Depersonalized: The majority of the pupils in the research stated that their mathematics learning was not personalized, but that it could be.

Goodall and colleagues (2017) suggested ALIVE concepts to increase mathematics performance in opposition to this TIRED mathematics:

- Accessibility principle refers to the use of appropriate enactive tasks that allow pupils to develop their own knowledge. The concept is proposed in the premise that these activities provide pupils with practically no justification to be excluded from developing mathematical thinking.
- Linked principle entails referring to prior knowledge in order for pupils to connect new material to what they already know and understand.
- Inclusive principle implies that all students participate in the process of learning mathematics through numerous activities, rather than believing that only extraordinarily bright children can learn mathematics.
- Valued principle stresses the use of real-life examples in the classroom in order to comprehend the worth of mathematics. The researchers went on to say that the examples should be drawn from those that individuals see as valuable both individually and culturally.
- Empowering principle relates to the students' agency, which allows them to take charge of their own learning. The goal is to assist students to have a better grasp of lifelong learning while making as much progress as possible in school mathematics. This principle also provides ways for students to be more empowered in their learning through carefully planned mathematics exercises, while also acquiring 21st-century abilities like creativity and technological literacy.

When establishing the RETA principles for geometry education, this design-based research was inspired by these frameworks in order to meet the needs in the common mathematics teaching approach of geometry in Turkey.

Common Mathematics Teaching Approach of Geometry in Turkey

Exam-Focused Pedagogy

Many studies including Bayart et al. (2000), Duval (1998), and Fujita et al. (2017) discovered and documented difficulties in 2D representations of various 3D shapes. These studies revealed that students performed poorly on these representations, particularly when constructing orthogonal and isometric drawings. When the reasons for poor performance on such drawings were investigated, the reviewed literature showed teachers as one of the most influential factors. That is to say, the literature indicated that teachers' beliefs influence their actions in teaching. Particularly mathematics teachers in Turkey believed that students' high national exam scores indicate teacher quality; and hence, teachers "teach to the test," and avoid student-centred activities (Doruk, 2014; Saralar-Aras, 2022). There was a lack of teacher incentive to teach 3D shapes without exam-focused teaching. This exam-focused pedagogy might be a result of continuing teacher performance evaluations in Turkey, where teachers' performance was primarily assessed based on student achievement on national tests since 2016 (Konan & Yilmaz, 2017, 2018).

Limited Use of Technology

Furthermore, in various studies (e.g., Bayrakdar-Çiftçi et al., 2013; Ciftci & Tatar, 2015; Tekalmaz, 2019), Turkish mathematics teachers were observed using the available technology in restricted ways, mostly to present recorded educational videos and quizzes from the ministry's Moodle page while wrapping up their sessions. Despite the emphasis on integrating technology into the new mathematics curriculum (Ministry of National Education [MoNE], 2018), particularly the use of dynamic geometry software packages in mathematics lessons, none of the observed Turkish teachers in Saralar's (2020) study encouraged students to use such software to visualize 2D representations of polycubes in any of their lessons on 3D shapes. In this study, the observed teachers stated that they believed in the present mathematics program's efficacy, and support its reliance on technology. When it came to teaching, however, most teachers believed that memorizing their techniques was the only way to master 2D representations of 3D shapes, and they recommended practising and drilling with a larger number of questions for better and faster results, rather than learning with and through technology.

Teacher-Centered Lessons

Last but not least, lessons on teaching polycubes were found to be teacher-centred in many studies despite the emphasis on student-centred activities in primary and secondary education programmes in the Turkish education system (Birgin & Acar, 2020; MoNE, 2018). Studies on this issue discovered that mathematics teachers dominated the use of tools and manipulatives (Saralar, 2020). In these courses, children were given very limited chances to utilize manipulatives and express themselves. Earlier research (Christou et al., 2006; Widder & Gorsky, 2013) suggests that this pedagogy is connected to the requirement to perceive 3D shapes from their 2D representations since it creates hurdles for not only students' learning but also teachers' teaching. According to numerous researchers, teaching 3D shapes in middle schools is regarded to be difficult among instructors, and as a result, teachers either do not teach it or utilize direct instruction rather than student-centred activities (e.g., Bakó, 2003).

Based on these findings and wider research, a model was created that proposes mathematics education, particularly teaching of geometry, may be realistic, exploratory, technology-enhanced, and active, hence the RETA principles. These principles led to the development of four lessons on orthogonal and isometric polycube drawings, one of which is available in Appendix 1.

METHODS

How the Work on this Approach Led to the RETA Model

The initial goal in developing the RETA model was to examine how national exam performance may be enhanced without relying on direct instruction or the repetition of national exam questions. Based on the wider literature and the findings of Saralar's (2020) study the RETA model was developed as an alternative to exam-focused pedagogy, limited use of technology and teacher-centred lessons. The following section briefly summarizes the RETA model, whilst explaining how the RETA model differs from more common approaches to teaching in Turkey.

First of all, rather than emphasizing the importance of the topic with the national exam questions, the RETA-based lessons were designed using real-life examples (R) and contexts. Secondly, the courses were supplemented with worked examples that were quite similar to national exam questions, but instead of drill and practice, they were meant to include mistakes for students to detect and remediate, which highlights the exploratory (E) principle. Thirdly, given that the Turkish government provided tablets to all students in the sample and interactive whiteboards to all classrooms, it was decided to strategically integrate dynamic mathematics tools such as GeoGebra and Cabri in the study of polycubes to provide multiple visual and dynamic representations using the technology-enhanced (T) principle. Finally, in response to teacher-centred teaching and teachers' dominance in manipulative use, the RETA's active (A) principle emphasized learning environments in which students have autonomy over manipulative use rather than passively following teacher constructions and replicating drawings on the board; hence, active refers to students' involvement as active participants.

RETA MODEL

Realistic (R), exploratory (E), technology-enhanced (T), and active (A) are the four principles of the RETA model. Even though these terms are polysemic, each principle is described in the context of the model, and the meanings connected with these principles are explained in the following sections. This section describes these principles more explicitly for their use in the context of the RETA model. For each principle, positive and negative aspects, as well as sample lesson applications, are discussed.

Realistic

Being the first principle of the RETA model, *realistic* lessons refer to the inclusion of real-life examples and situations in the teaching. Real-life examples demonstrate how the knowledge and skills gained in the mathematics course may be used in the real world (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2003). This is meant to raise students' awareness of the topic's relevance in their everyday lives, allow them to draw inferences about mathematical ideas' real-world connections, increase motivation for studying mathematics (three-dimensional shapes in this specific case), and assist students in solving real-world issues in the future. The immediate goal of this research was to improve on the Turkish government's national geometry exam, however, it is hoped that there is more to it than just passing the Turkish national exam. The following examines the spatial and mathematical aspects (importance) of realistic lessons before explaining how they are applied in the sample lessons prepared.

When we think of how three-dimensional shapes are related to real life, spatial thinking comes into conversation as the first aspect. Spatial thinking is employed in a variety of ways in everyday life. To illustrate, we utilize spatial thinking automatically while preparing luggage and packing it into a car trunk (Liben, 2007). It is also required for education programs, ranging from the use of molecular models in chemistry (Barke & Engida, 2001; Pribyl & Bodner, 2006) to the comprehension of mountain strata in geography (Lee & Bednarz, 2009; Robertson et al., 2009). Spatial thinking is also seen as a vital skill in many disciplines including science, engineering, and mathematics, particularly in geometry.

Many occupations, from radiologists to product designers, require the ability to think in two and three dimensions. Doctors, for example, study two-dimensional scans of the body generated by radiologists to identify the patient's problems. Architects who sketch their construction designs on paper or on a computer screen to depict them in two dimensions are another and a closer approximation to geometrical drawings. They create architectural plans and elevations to reduce three-dimensional forms into a sequence of two-dimensional components for a variety of purposes, including solar energy conservation, and heating and electrical structures (Matusiak, 2017). Elevations are crucial to understand key dimensions such as wall lengths and heights, as well as to show openings such as doors and windows; and, plans are handy depictions to understand proximity and spatial relations between the rooms of a building, thus, they are both useful in real-life contexts. In middle school geometry curricula, plans are mostly called top views, while elevations are termed views from the front, rear/back, left, and right. The author designed lessons on teaching elevations. Incorporating such real-life examples into lesson plans might pique students' attention and motivate them to study more about the subject (Fredricks et al., 2017).

The second (mathematical) aspect of realistic education of the RETA model considers English realistic mathematics education as an example. That is to say, the author took the studies that are conducted in the United Kingdom into consideration with the knowledge that it is one of the first countries to embrace and implement realistic mathematics education (De Lange, 1996) – which started early in the 1970s to enhance the quality of mathematics instruction in Dutch schools (Freudenthal, 1971, 1973). According to Freudenthal (1987), there is a global need to apply mathematics, and realistic mathematics education is one approach to do so. According to him, by starting and staying in reality while teaching mathematics, realistic mathematics education creates mathematics of human worth. It relates mathematics to reality in order to keep the children's attention and encourage them to study mathematics (van den Heuvel-Panhuizen, 2003).

It is much easier for children to make this connection (subject and its real-life relation) when it is more noticeable as in the study of population in geography or the study of motion of objects in physics. However, many children and even adults struggle to make this connection when it comes to mathematical topics (Cornell, 1999; Larkin & Jorgensen, 2016). They either see mathematics as an abstract discipline or only link specific aspects of mathematics to real life, but they fail to explain the broader application of these elements in actual life (e.g., Mulero et al., 2013; Reid et al., 2003). For instance, Mulero and friends (2013) showed that more than half of 94 university students did not associate architecture with mathematics and did not even know any architect who is pioneering a mathematical contribution to architecture. While this is the situation, it is critical to give a realistic mathematics education that allows children to recognize the connection between mathematics and real life as early as elementary school.

Children learn mathematics based on tasks they may face in their daily lives in *realistic mathematics education*, and they have the chance to create their own knowledge via group work, debate, and reflection (van den Heuvel-Panhuizen & Drijvers, 2014). These criteria are consistent with constructivist theories (Cobb, 1994; Cobb & Yackel, 1995; Gravemeijer, 1994; Simon, 1995) and correspond to the RETA model's other principles, such as exploratory and active, as detailed in the following sections.

An excellent example of an effective intervention that complements the author's approach to teaching mathematics is English realistic mathematics education, especially geometry (Cooper & Harries, 2009; Dickinson & Eade, 2005; Dickinson & Hough, 2012). For example, in 2003, a local Manchester school trialled realistic mathematics education in English classes with Year 7 children, and the response to the created realistic education materials was extremely favourable (Dickinson & Eade, 2005; Eade & Dickinson, 2006). This study took place in over 20 schools, and data revealed that children's comprehension of mathematics and approach to problem-solving improved. Furthermore, not only did students who actively engaged with the material improve but so did lower attaining ones. Over the course of three years, there have been consistent findings concerning increases in mathematics proficiency in different areas, including geometry in mathematics (Blum et al., 2019; Dickinson et al., 2010; Hough et al., 2017; van den Heuvel-Panhuizen, 2019). Since then, this realistic approach to mathematics instruction has been implemented into middle school mathematics classes in the United Kingdom (Dickinson & Hough, 2012), and it has served as the foundation for the realistic principle of the RETA Model.

Together with all the positive responses, there are also several criticisms of realistic mathematics education. To begin with, there is a criticism that it has been exaggerated, and that abstract mathematical concepts are far superior to realism (Keune, 1998, 1999). Regarding this, in realistic mathematics education, "there is a need for giving more attention to abstraction and logical reasoning to better make use of the Dutch mathematical talent that would be lost because of the realistic approach," said Keune (1999, p. 365). However, it is essential to highlight that Keune's (1999) speech, which outlines his perspective on realistic mathematics education, focused on what realistic mathematics education has become in the Netherlands rather than what it is meant to be. As a response to this criticism, the author agrees with Gemert (2015) that just because something is practical does not mean it cannot be abstracted or taught through examples. Considering this, in addition to including realistic mathematics, the RETA model included worked examples for practice, explained in the Exploratory principle.

Secondly, realistic mathematics education has been criticized as being reductionist since it gives realistic but not actual settings (Verstappen, 1994). According to Verstappen (1994), issues in realistic mathematics education give simplified actual situations that may subsequently pose difficulty in managing real problems formally in mathematics and later in life. Gravemeijer (2001) reacted to this by highlighting that the issues in realistic mathematics education can, but do not have to, deal with actual everyday life situations that are more difficult for children and involve many variables than what realistic mathematics education recommends. According to him, providing a familiar setting in which students can act intelligently in order to fully understand the mathematics in it is critical in realistic mathematics.

Finally, some studies demonstrate that just presenting things in a realistic setting does not always make things simpler to understand for children. For instance, Chandler and Kamii's (2009) experiment with 98 children revealed that using coins as a real-life example to teach place value makes it more difficult for children to grasp the topic. They discovered that while it is easy for teachers to conceive of one dime as ten pennies as adults, it is difficult for children to do so, and it is considerably more difficult when teachers incorporate monetary examples in the teaching of ones and tens. This is just one example showing that real world applications may make teaching certain concepts more difficult, thus real-life examples should be carefully chosen to meet the needs of the children. Considering both advantages and disadvantages, the next paragraph discusses how to include realistic mathematics into the sample lessons.

In the designed lessons, videos and photos are utilized to bring the concepts of orthogonal and isometric drawing of polycubes to life (see Appendix 1 for a sample lesson). Many studies recommend real-life videos and explanations showing mathematical material in action. However, some critics have claimed that videos are perceived by children as entertainment rather than information, resulting in children not benefitting from the videos as much as they could (Salomon & Perkins, 2005). As a result, in the sample RETA-based lessons, the design decision was made to not only present important material in the videos but also to create a student-centred atmosphere for discussion. Peer discussions, as well as a subsequent whole-class discussion, were meant to assist students to establish links between classroom knowledge and skills and their real-life applications. Furthermore, nearly every unit cube construction made by children is a depiction of a realistic image of a building (e.g., a castle and a school).

Exploratory

The term *exploratory* describes the second principle, which relates to the use of worked examples in classes to assist students in investigating the topic. Worked examples are pedagogical devices that give students someone else's answer to study (Evans & Swan, 2014). The exploratory principle is aimed at providing students with worked examples of the topic with some of these examples designed to include mistakes to explore and remediate. These mistakes could be chosen from the most common mistakes of the students, reported by available research. The aim for including these worked examples is, in a way, taking advantage of problem-solving opportunities, as many argue that problem-solving affords exploration (e.g., Carreira & Jacinto, 2019; Schoenfeld, 1985, 2013), and students who study geometry through exploration with problem-solving achieve higher scores than those who study with traditional methods in middle school geometry (see Klančar et al., 2021).

Geometry is mostly taught via examples, which have a place in many teaching and learning theories (Bruner, 2017; Marton et al., 2013; Marton et al., 2004; Skemp, 2012; Watson et al., 2006; Wilson, 1986). Students are frequently asked to work on examples and solve problems in geometry classes. Solving problems, nonetheless, may not be particularly useful when children are just beginning to study and have just basic knowledge of the subject (Renkl, 2011; Salden et al., 2009). When compared to solving problems at the beginning of the lessons, worked examples were shown to be more successful for initial skill acquisition (Kalyuga et al., 2001; Renkl, 2014, 2017). Hence, in the design of lessons with the exploratory principle of the RETA, it may be suggested to include worked examples when introducing the topic.

Moreover, learning geometry through criticising, comparing, and debating various solutions has several advantages, ranging from increasing student engagement with examples to supporting students in successfully integrating previous knowledge into present learning processes (Pierce et al.,, 2011; Silver et al., 2014). "Mathematical discourse has long been shown influential in supporting students' learning of mathematics" (Bennett, 2010, p.79). Many mathematics education researchers support the idea that dialogic teaching is beneficial (Bakker et al., 2015; Hofmann & Mercer, 2016; Kazak et al., 2015; Mercer & Sams, 2006; Ruthven et al., 2017; Warwick et al., 2016). Some also argue that dialogic processes aid conceptual development in mathematics (e.g., Kazak et al., 2015). Yet, dialogue during work with examples is not without its criticisms. Some researchers have argued that implementing dialogic mathematics teaching might be difficult, not just for newly qualified teachers (Bennett, 2010) but also for more experienced ones (Wegerif & Scrimshaw, 1997). More recent research also found evidence indicating that not only teachers but also students, frequently oppose discourse in a mathematics lesson. Regarding this, Bennett (2010) asserted that "it's hard getting kids to talk about math" (p.79). Particularly, in the country of Turkey, except for occasional experiments for research purposes (e.g., Gürbüz & Agsu, 2017), dialogic mathematics education is not a prevalent teaching approach. Teachers of mathematics are still lecturing in front of the class, and there is little room for student discussion and interaction in mathematics classes in Turkey; therefore, there may be resistance to dialogic teaching.

In RETA-based lesson plans, similar to Durkin and Rittle-Johnson (2012) and Evans and Swan (2014), some of the planned worked examples have intentionally designed mistakes for students to diagnose, remediate, and debate probable causes for them. This is trialled previously and found to enhance students' understanding of geometry topics. For example, Evans and Swan (2014) worked with eight secondary school mathematics teachers in the United Kingdom and over twenty in the United States to integrate worked examples with intentional errors into their lessons. The teachers prepared lessons using the worked examples provided by the researchers. These examples are available at http://map.mathshell.org/ on the Mathematics Assessment Project's official website. When the children failed to solve a geometry problem, these teachers offered them to work with worked examples. The teachers highlighted as a constraint that some children were more concerned with correcting mistakes than making holistic comparisons. Nonetheless, all students managed to improve their performance on the mathematics test. This could be related to the productive failure of Kapur (2014) and Sinha and Kapur (2021) who argue that failure in problem-solving, particularly when followed by discussion, is productive and results in learning (Hedge's $g = 0.87$).

To help students develop their conceptual understanding of holistic concerns related to the topic (2D representations of polycubes), the designed lessons featured peer discussions followed by a whole-class discussion. The author was aware that conversations were a) more demanding for both teachers and students due to the abilities required to evaluate and analyse the thinking behind solutions, and b) significantly different from how teaching takes place in the Turkish setting. To conclude, the model's second principle attempts to provide an exploratory mathematical education in which students engage with worked examples, questions, and mathematics dialogue.

Technology-Enhanced

The third principle advocates for *technology-enhanced* learning, which refers to the strategic use of dynamic tools in teaching mathematics. There may be various dynamic geometry tools to integrate into mathematics lessons including Cabri, GeoGebra and SketchPad. The aim of integrating these tools into the mathematics lessons through the technology-enhanced principle is to present various visual and dynamic representations of mathematics topics. The following examines two elements of technology-enhanced education: spatial and mathematical aspects, as well as their implementations in the designed sample lessons.

People many find it challenging to reason about three-dimensional shapes when working with two-dimensional representations, thus making this a more complex task (Reisberg & Heuer, 2005). In order to impart meaning to three-dimensional shapes, people frequently create two-dimensional representations. Two-dimensional representations can be incorporated into the human mind in a variety of ways, such as using landmarks based on the properties of a shape or by reference to other shapes (Tversky, 2005). Reisberg and Heuer (2005) point out that "mental images seem to be represented from a determinate viewing angle and distance … since visibility from a perspective and occlusion seem to play a role in those data" (p.39). This statement is supported by their depiction study, which shows that when people are asked to describe a cat in a picture, their responses are faster than when they are asked to describe specific parts of a cat, and that they are even faster to describe large and visible parts like the cat's head in the picture than small and hidden parts like whiskers and claws. Cubes that lie behind other cubes in geometry courses on polycubical shapes may be partially visible or invisible, making it difficult to express them in two-dimensional forms, such as orthogonally or isometrically.

Although numerous studies have shown that dynamic geometry tools can assist students in expressing three-dimensional structures in two dimensions (e.g., Oldknow & Tetlow, 2008; Simpson et al., 2006), the introduction of technology to geometry instruction has historically been met with opposition (Bolt, 1991). The Royal Society's working group on teaching and learning geometry (2001) advised that students pay more attention to studying three-dimensional shapes and that they learn them more effectively through the use of digital tools (such as virtual manipulatives and dynamic geometry tools) in the classroom. Taking this into account, Oldknow and Tetlow (2008) tested the efficiency of a 3D geometry program in small-scale pilot schools before expanding their research to a larger project in a set of Hampshire schools in the United Kingdom. Their research found that working with such software — which allows students to build two-dimensional representations of three-dimensional shapes — gives students a deeper understanding of three-dimensional shapes as well as a wealth of opportunities for active participation, collaboration, and confidence-building. According to Widder and Gorsky (2013), who investigated students' utilization of mathematics software in three-dimensional shape lessons, students utilized these tools according to their needs. That is to say, students with different pre-test scores used the software for different purposes: those with low spatial skills used it primarily for measuring their constructed representations, while those with high spatial abilities used it primarily for self-examination (e.g., 50 per cent more constructions and operations than their peers with lower spatial skills) and for speeding up mental processes like (re)constructing and rotating.

Despite the fact that all representations have benefits and drawbacks (Friedlander & Tabach, 2001), many researchers agree that learning from an appropriate combination of representations with the help of technology is more beneficial than learning from a single representation (e.g., single-use of verbal, numeric, symbolic, or graphical representations), and geometry software packages provide an environment where this can happen (Ainsworth, 2006; Hoyles & Noss, 2003; Kaput, 1992; Pape & Tchoshanov, 2001; Pierce et al., 2011). However, not all of the results/applications of incorporating dynamic technologies are beneficial. Lessons with these technologies are more difficult to plan (Grandgenett, 2007), especially since they demand the use of more student-centred approaches, and they are more challenging for teachers to manage in general (Bates, 2005). Furthermore, a lack of teacher knowledge in teaching geometry with technology, as well as inconsistent beliefs and goals regarding the use of technology, make it much more difficult to develop technology-enhanced lessons. Teachers' knowledge, beliefs, and goals all influence how they teach with technology (Ball et al., 2008; Mishra & Koehler, 2006; Niess, 2008, Shulman, 1986). While some teachers regard dynamic geometry tools as distractions rather than learning aids and find them time-consuming, others see them as a highly useful facilitator and successful teaching method (Saralar & Ainsworth, 2017; Saralar-Aras, 2022).

In the study of sample lessons, teaching was strategically supported with GeoGebra, free dynamic software for manipulating 3D shapes that may be used individually on tablets or collectively on interactive boards. This decision was partially pragmatic, as 1.5 million tablets were handed to children and all middle school classes in Turkey were equipped with interactive whiteboards from 2011. To conclude, by using dynamic geometry software, the third principle of this approach intends to provide students with *technology-enhanced* experiences of an acceptable combination of 2D representations of 3D shapes.

Active

The fourth principle relates to *active* learning settings in which students manage the use of concrete manipulatives rather than watching teacher constructs and reproducing drawings on the board, as seen in various studies (e.g., Saralar, 2020). Active lessons in the RETA model are more than merely a contrast to passive learning of Chi (2009) and Schank (1994). Although the activities involved could be described as constructive (e.g., students building cube constructions for themselves) and sometimes interactive (e.g., looking at students' cube-constructions and solutions and receiving feedback on them) in the literature (see Chi, 2009), it was chosen to use the term active because this principle refers to the involvement of learners as active participants. With the active principle, students are intended to engage in the learning process through the control of manipulatives. This is intended to increase students' motivation as well as their academic performance in the intended topics. The next paragraphs detail how concrete manipulatives were used to make students active, review assertions about their use and advantages in geometry, and examine how they were used in the sample lessons.

To deliver middle school mathematics in the past, teachers depended on workbooks and memory. Nevertheless, educators have argued for over two decades that these techniques are "ineffective and outdated" (Cain-Caston, 1996, p.271). Many researchers who compared standard teaching techniques to alternative methods, such as concrete manipulatives, discovered that children perform better with alternative methods (An & Tillman, 2015; Driscoll, 1983; Kong & Mohd-Matore, 2022; Suydam, 1984). Concrete manipulatives are "objects that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered" (Swan & Marshall, 2010, p.14). Concrete manipulatives are also previously defined as "models that incorporate mathematical concepts, appeal to several senses and can be touched and moved around by students", which incorporates the author's notion of making students active (Hynes, 1986, p.11). Unit cubes, such as multilink cubes or linking cubes, and unifix cubes are referred to as concrete manipulatives in the sample lesson plans.

In the literature, there is no unanimity on the use of concrete manipulatives. While some researchers argue against manipulatives (e.g., Ross, 1989; Uttal et al., 1997), others argue in favour of them (e.g., Moch, 2001; Van de Walle et al., 2010).

The author agrees with proponents who believe based on their empirical study that manipulatives work if specific circumstances are met. Plastics cannot teach mathematics, as Ball (1992) emphasizes with the following words: "Understanding does not travel through the fingertips and the arm… Mathematical ideas really do not reside in cardboard and plastic materials." (p.47). Using manipulatives does not help students learn new mathematics quickly. Many academics have identified specific circumstances that make manipulatives helpful, such as effective lesson plans and students' desire to learn meaningfully (Carbonneau et al., 2013; Furner & Worrell, 2017; Moch, 2001). Instruction, for example, is stressed as a critical element. The way teachers teach with manipulatives determines how effective manipulatives are. Carbonneau and colleagues (2013) conducted a meta-analysis using 7237 students from 55 studies to determine the usefulness of teaching mathematics with concrete manipulatives. Their results showed that manipulatives offer greater learning than standard approaches that simply supply abstract mathematical symbols. Instruction was shown to regulate the connection between students' learning and their use of concrete manipulatives. It also depends on students' understanding of what manipulatives represent, and on the teacher's directions and assistance throughout the classroom exercise (Uttal et al., 1997). According to theories in the literature, concrete manipulatives aid learning when they promote any of the following instructional characteristics:

- Providing chances for student-centered research on the subject (Kirschner et al., 2006; Mayer, 2004),
- Assisting students in their abstract reasoning (Bruner, 1964; Montessori, 1964; Piaget, 1962),
- Providing physical enactment (Biazak et al., 2010; Engelkamp et al., 1994; Kormi-Nouri et al., 1994),
- Encouraging students to apply what they've learned in class to real-life situations (Baranes et al., 1989; Rittle-Johnson & Koedinger, 2005; Tindall-Ford & Sweller, 2006).

Environments in which manipulatives are utilized for meaningful learning by building on prior knowledge and challenging students to reflect and think are another example of these circumstances (Baroody, 1989; Furner & Worrell, 2017). Both Baroody (1989) and Furner and Worrell (2017) stress that manipulatives aid students when they use inquiry to connect their prior knowledge to the desired learning goals. When students have prior experience with the subject (Sowell, 1989) and use it regularly over time (Marzolf & DeLoache, 1994), the use of manipulatives in the classroom leads to even higher learning results.

Teachers frequently use unit cubes to teach three-dimensional shapes in geometry classes. According to Swan and Marshall's (2010) study of 249 teachers from New South Wales, cubes are the most commonly used manipulative in introducing three-dimensional shapes and the third most commonly used manipulative in mathematics classes (after blocks and polyhedrons) followed by unifix cubes and multilink cubes. Although there are pros and cons to utilizing these manipulatives, many researchers agree that they are helpful for teaching and learning mathematics and that they improve students' understanding of mathematics by enabling them to explore the topic actively (e.g., Canny, 1984; Clements & Battista, 1990; Skemp, 1987; Suydam, 1984). Children's use of concrete manipulatives can help them visualize shapes better and so improve their mathematics understanding. Regarding this, "The relevant application of manipulatives to … classroom situations helps students visualize and develop problem-solving strategies", says Moch (2001, p.83). Furthermore, concrete manipulatives, particularly cubes, can help children acquire more meaningful mathematical thinking and reasoning by allowing them to build and compare amounts, and allowing them to create interlinked understandings of mathematical ideas (Stein & Bovalino, 2001). Stated differently, via their experience with concrete manipulatives, children may integrate and link a range of concepts and develop a profound knowledge of them. Concrete manipulatives may also give students concrete and exploratory experiences for two-dimensional representations, allowing them to embody the problem scenario by touching, manipulating, and determining the proper structure (e.g., see Carroll & Porter, 1997).

However, if teachers monopolize the use of these manipulatives and students are not given opportunities to actively engage with them by touching and moving them around, teaching polycubes with them may not be particularly useful. The way teachers use concrete manipulatives in their lessons on polycubes, as well as how they teach other mathematics concepts, is crucial to students' success (Alfieri et al., 2011; Wearne & Hiebert, 1988). Saralar (2020) observed that teachers dominated the use of manipulatives in all observed Turkish classes, and as a result, children appeared to be disengaged with the instructional material. This might be a major factor in children's poor performance on three-dimensional shape comprehension tests. Three-dimensional shapes education must become more student-centred, with classrooms where students actively engage in rich mathematical activities during which they have control over manipulatives, to boost students' engagement and improve their knowledge of shapes.

Furthermore, the disparity in the usage of manipulatives in the literature indicated a very particular rationale for why they might be useful in the situations investigated in this study in Turkey. Most people who argue against manipulatives utilize blocks to represent abstract ideas, such as place value and percentages, which are represented by addition and subtraction (e.g., Bartolini-Bussi & Mariotti, 2002; Chandler & Kamii, 2009; Fuson & Briars, 1990). The active principle of the RETA model, on the other hand, is quite similar to how tangible manipulatives are employed in chemistry. Students in chemistry create a model containing atoms and structures so that they do not have to continually envision 3D structures; instead, they externalize these into a 3D model (see Hegarty et al., 2013). In a similar vein, in the sample RETA-based lessons, students construct polycubes from concrete unit cubes to externalize them so that they don't have to envision them while drawing orthogonal and isometric views. Constructions will not reflect anything else in mathematics (for example, abstract symbols), but will simply be externalized to allow students to draw them themselves. As a result, the researcher's goal in utilizing manipulatives is to teach symbolic relations rather than traditional mathematical facts. To summarize, active lessons strive to create learner-centered environments in which students interact with concrete manipulatives. The designed lessons (one of which is in Appendix 1) are meant to allow students to explore polycubical forms not just via student-centred activities with unit cubes but also through chances for reflection through teacher-led conversations.

SOLUTIONS AND RECOMMENDATIONS

The RETA model's principles may be applied to a wide range of topics in mathematics. The work on this has already started, and Turkish mathematics teachers started to prepare RETA-based lesson plans for various topics (other than 3D shapes' orthogonal and isometric drawings) in mathematics from polygons to types of triangles (e.g., Esen & Saralar-Aras, 2021). Other teachers and/or teacher-researchers than the author could continue designing RETA-based lessons and evaluate the effectiveness of these lessons by measuring students' academic performance through various types of assessments.

The RETA principles are not claimed to be sufficient; additional principles can be created or implemented, but the presented principles are required for teaching three-dimensional shapes in the studied context in Turkey. Furthermore, lesson plans do not have to employ all of the model's principles in a single lesson plan; rather, they may use them in their overall teaching of the objectives from mathematics.

Moreover, such research in real classrooms can provide valuable insights for teachers, educators, programme developers and policymakers. Teachers may gain self-awareness about their practices while applying RETA principles, whilst educators can use and trial the model in other contexts to see whether it meets the needs of their specific contexts. Finally, policymakers and programme developers can benefit from the findings. In particular, the Turkish Ministry of National Education and programme developers working for the ministry might be interested in the study as the sample lesson plans were developed specifically for the Turkish context.

FUTURE RESEARCH DIRECTIONS

Each classroom has a different student profile, particular traditions and contextual needs with its own atmosphere. While there are various ways of teaching mathematics, there can be particular ways that work for specific contexts. The RETA model was developed for the Turkish context, and sample lesson plans were developed and found to be effective in teaching orthogonal and isometric drawings of polycubes. In a quasi-experimental study with more than 200 students, scores of the RETA intervention classes improved almost up to 100% in orthogonal drawings and almost 90% in isometric drawings that are equivalent to national exam, whilst the final performance was about 60% in orthogonal drawings and 45% in isometric drawings in the traditional classes (Saralar, 2020; published elsewhere). Although it was developed as a response to the need in the Turkish context, clearly, there are various ways that the need could be met. This model is only one of many possibilities that might have been developed.

The chapter outlined the need for student-centred environments, the need to motivate teachers to teach 3D shapes, the need to integrate realistic scenarios to increase student engagement, and the need to provide technologies that can help both students' learning and teachers' teaching. Therefore, the RETA was created to address the specific needs of middle school children in the Turkish context. Overall, the results show that RETA-based lessons helped students score better on the national exam. However, this does not imply that the RETA model and RETA-based teachings are flawless, prompting the issue of how the RETA model and RETA-based lessons may be improved. For example, for future studies, it is possible to supplement other principles to the RETA principles according to the needs in other contexts. For example, in retrospect, the author wonders why, as a researcher who appreciates conversation in her classes, she didn't explicitly list dialogue as a principle. It is suspected that one explanation is as a Turkish mathematics teacher herself, she resisted having it as a principle while knowing it would be challenging in the Turkish context.

Last but not least, there is considerable scope for this model to be tested and further developed in a range of contexts. The RETA model is trialled as a quasi-experiment only for one topic, orthogonal and isometric drawings of polycubes, which is from a middle school programme in Turkey. It could be utilized for other topics, as well as other levels of education. There is a much greater geometric world that the RETA model could be instantiated, in other years in middle schools, and other towns and countries. Moreover, in addition to the RETA model itself, the RETA-based lesson plans can be tested and improved. Having a greater sample of teachers that have met specific criteria such as years of experience and background of teaching would have been ideal when testing RETA-based lessons.

CONCLUSION

To conclude, study on students' geometry proficiency has long been considered an important research issue in mathematics (Clements, 2003; Clements & Battista, 1992). This chapter introduced a reform-based mathematics teaching model, the RETA model, as well as design decisions made during its application in sample lessons to teach geometry objectives. It is anticipated that this model, as well as the lesson plans that were developed utilizing the RETA principles, would be useful in the future for teachers teaching in Turkey, and maybe beyond.

To note, the chapter was derived from the author's PhD thesis titled "Designing Lessons to Help Middle School Students Learn about Orthogonal and Isometric Drawings of Three-dimensional Shapes".

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KEY TERMS AND DEFINITIONS

Isometric Drawing: A two-dimensional representation of a 3D object with three primary lines which are angled away from the observer in the same way with vertical lines staying vertical whereas horizontal lines at 30-degree angle to the horizontal plane. Isometric projections; perspective drawings.

Orthogonal Drawing: A two-dimensional representation of a 3D object from the top, bottom, front, back, left and right. Orthogonal projections; orthographic projections/drawings; plan/top view, and elevations /side views.

Polycube: A three-dimensional shape which is constructed from unit cubes.

RETA Model: A model for teaching mathematics that supports realistic, exploratory, technology-enhanced, and active learning.

APPENDIX

The process of designing the following lesson plan was based on the RETA model and the reviewed literature. Turkish mathematics teachers had a very tight curriculum to cover with a fixed time for teaching the topic. It should also be noted that the lesson was designed to meet the needs of the children in the sample and does not claim to be the ideal option for different educational settings. The researcher developed the design decisions based on the RETA model and her ability to construct lesson plans as a mathematics teacher and a researcher. As a result, other researchers would make various decisions to meet the needs of the students in their classrooms; yet, the decisions were able to meet the needs of the current teaching in Turkey.

Each lesson plan comprised a lesson abstract, a lesson structure that specified the amount of time allotted for each activity, and activity descriptions in the form of a teacher's guide. It is worth noting that different classrooms had varied dynamics and student profiles, making it difficult to forecast how the lesson would proceed in different classes, such as how students would respond to a specific activity. As a result, the mathematics teachers who used these plans did not carefully adhere to the time allotted for each task. The first of the four developed lesson plans to teach 2D representations of 3D shapes follows.

Sample Lesson Plan

Lesson abstract: Students focus on the issue of why we need two-dimensional drawings (orthogonal and isometric drawings) of three-dimensional objects. They develop an awareness of how drawing views from the top, front and sides and isometric drawing are related to real-life practices. They engage with several real-life examples and consider how these may be represented mathematically (realistic). After engaging with the real-life examples, they construct polycubical shapes corresponding to pictures of 3D objects (realistic) with linking cubes (active). They create their concrete polycubical shapes in the authoring tool, created through GeoGebra. They explore the view from the front by manipulating their representation in GeoGebra (technology-enhanced). They develop an awareness of how the front views change when they manipulate the shape. They diagnose and remediate and discuss possible reasons for worked examples with designed mistakes (exploratory). Lesson 1 focused particularly on a realistic principle of the RETA model where students articulate real-world experiences through videos and photographs on engineering, architecture and a drawing tool AutoCAD.

Lesson Structure

- Agenda and aims (5 minutes)
- Real-life examples: Engineering, Architecture (10 minutes)
- Tools for drawing: AutoCAD, GeoGebra (15 minutes)
- Activity 1: Constructing buildings and drawing their front views (15 minutes)
- Activity 2: Finding the mistakes (5 minutes)
- Conclusion and feedback (5 minutes)

Agenda and aims: Introduce yourself, if necessary. Explain to students that during the next four lessons we are going to discover different types of drawings and drawing tools, and relate our learning with their use in real-life situations. Ask whether students have any question before starting to the first lesson, and answer their questions if any. Explain to students that in this lesson we are going to consider some real-life examples to understand why people need two-dimensional representations of three-dimensional objects.

Real-life example 1 - Engineering: Show a two-minute part of a technical drawing video. Explain why we might need to learn such drawings and how these can be related to mathematics. Explain that this is how engineers start to draw multi-views of the shapes. They use different views to design different parts of the machines. All of the machines we use in daily life, such as computers, mobile phones, hairdryers and fruit squeezers, are composed of small parts, which were designed and drawn by engineers. Ask whether anyone's mother or father is an engineer or whether anyone has seen such a drawing before. Invite students to share their ideas as well.

Real-life example 2 - Architecture: Show the whole class an architecture photo and repeat the same procedure in Engineering activity. Discuss why architects need to learn these drawings and how these can be related to mathematics. Invite students to share their ideas as well. Summarize the discussion with a few sentences.

Explain that the drawing in the video is the first step to draw plans of the houses, or draw new interior designs of the houses. All architects similar to engineers learn how to draw multi-views and prepare projects based on this knowledge. Such drawings are natural parts of their jobs. Architects also try to draw these shapes more clearly as their drawing should be easy to understand by the people who ask for their help or their customers.

Tools for drawing 1 - AutoCAD: Show whole class a two-minute part of an AutoCAD drawing video. Explain that this is how engineers use a tool to construct their shapes.

Discuss how such tools might help them draw three-dimensional polycubical shapes in two-dimension. Invite students to share their ideas as well.

Possible Prompt Questions

- What are the potential advantages and disadvantages of using a tool like this?
- What about accuracy?
- What about getting the detail right?
- What about time spent on construction?
- Why do you think we both need tools and pen and pencil drawings? – A possible answer: Sometimes it is easier to sketch, sometimes tools help us to visualise the shape so that we can draw with pen and pencil.
- Do you think using a similar tool in this class help us visualise different views of a shape? How? – An expected answer: with the help of the manipulations the tool allows

Tools for drawing 2 - GeoGebra: Explain to students that there are some tools which help us in drawing shapes similar to engineers. Introduce the authoring tool. Ask students to turn on their tablets and run the authoring tool created through GeoGebra. Give them some time to explore how it works. Move to construction examples after the discovery of GeoGebra.

Open the slide – Exploring the Authoring Tool in Figure 1. Give an example (use a simple construction similar to the one on the slide) to show how buttons work and how we can manipulate the shape constructed on the authoring tool. Ask students to construct the same shape on their tools. Continue asking questions to stimulate their exploration of the tool.

Possible Guiding Questions

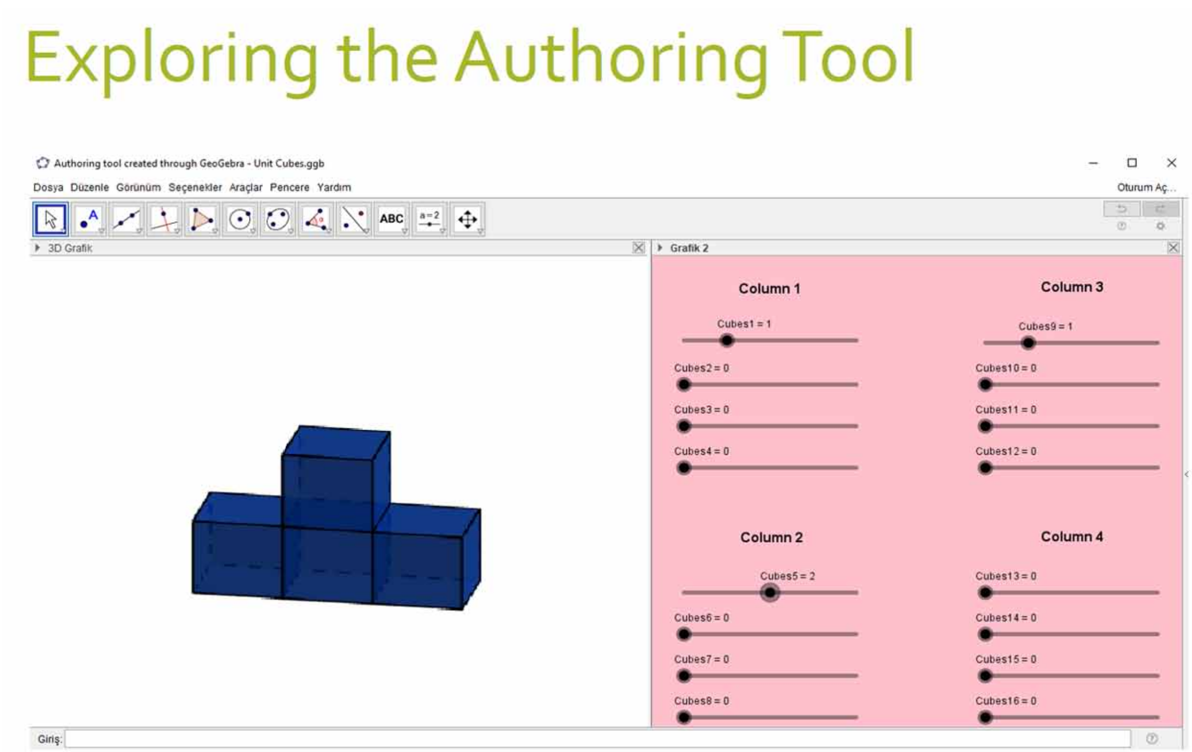
- 2 x 2 x 1 (Length x Breadth/Width x Height)
- 3 x 3 x 3

Then, ask them to remove cubes so that they have 3D shapes which have the following dimensions:

- 3 x 2 x 2
- 2 x 2 x 2
- 2 x 1 x 2

Move to construction examples after the discovery of GeoGebra.

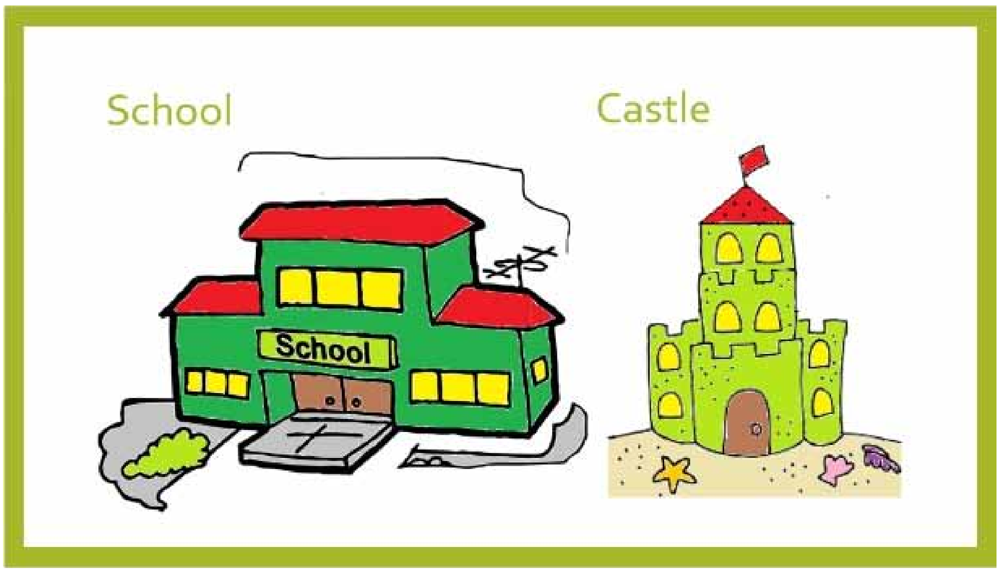
Figure 1. Exploring the authoring tool



Activity 1

Organize students into groups of 2-3. Give a box of linking cubes to each group. Show the school picture and ask them to construct the school using linking cubes. Check their constructions and discuss their answers. Repeat the same procedure for the castle. Possible correct answers are included in Figure 2.

Figure 2. School and castle pictures – I



Students may construct the castle totally different than each other as only one view (front view) of the castle is seen from the pictorial representation. Here, explain that we need more than one view to construct the exact shape. Then, ask at least how many views we need and why?

Figure 3. Possible correct constructions from linking cubes – I

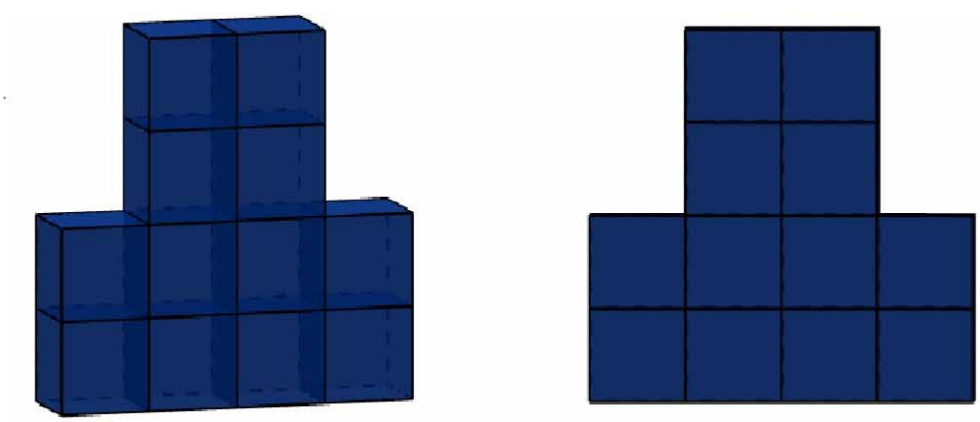
Point to the castle in Figure 2. Ask them to construct the castle from linking cubes. Following that ask them to construct the same castle on the tool (by now they are familiar with both the polycubical shape and its pictorial representation).

Note: If there are students who constructed with a different depth, remind the point discussed before: Having only one view of a shape is not enough to decide all of its dimensions, therefore we may not construct the exact shape only having its one view. Ask them to remind you at least how many views we need to construct the exact shape and why?

After they all construct the shapes on the authoring tool, give a copy of the Activity Sheet Castles and focus on the front view of the first representation. Invite the group of students to manipulate their GeoGebra constructions to decide how to represent the front view. Ask all students to draw the front view on the dotted paper individually.

Do not forget to ask students to save their files before moving to the next question on the sheet. Please note that it might be useful or easier for students to see the depth if we use the angles of the isometric paper on the tool (30^o-60^o), so before giving the authoring tool to students set the angle accordingly. After they manipulate the shape they may decide which angle they would like to use. Figure 4 shows a possible correct GeoGebra construction of the castle and its front view.

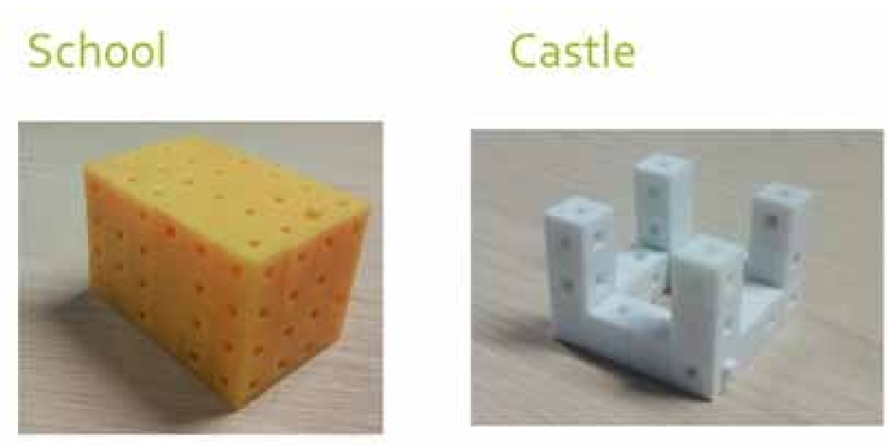
Figure 4. A possible correct GeoGebra construction and its front view – I



Show the next slide having pictures of the buildings which require relatively more complex constructions in Figure 5. Point to the school picture and ask them to construct the school using linking cubes or/and GeoGebra. Check their constructions and discuss their answers. Repeat the same procedure for the castle. Collect students' constructions to give them back in the next lesson. Some of the possible correct answers are presented in Figure 6.

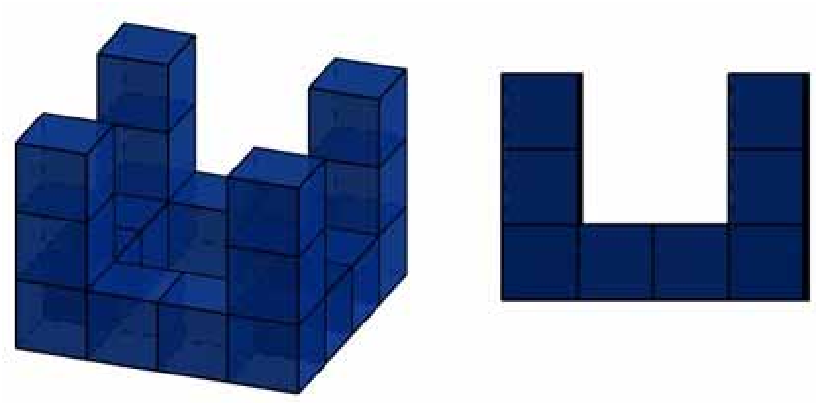
Figure 5. School and castle pictures – II

Figure 6. Possible correct constructions from linking cubes – II



Point to the Castle II in Figure 5. Ask students to construct the castle, which they constructed from the linking cubes, on the tool. After they all construct the shapes on the authoring tool, invite them to focus on the front view of their Castle II representation on the tool. Invite the groups of students to manipulate their GeoGebra constructions to decide how to represent the front view. Ask all students to draw the front view on the dotted paper individually. Figure 7 presents a possible correct construction of the castle and its front view.

Figure 7. A possible correct GeoGebra construction and its front view – II



Show the slide – Question – III in Figure 8. Follow the same procedure for the third question of the worksheet as well. This time, do not ask students to construct the shape with linking cubes. However, students who need concrete construction may continue constructing with them. Do not forget to ask students to save their files in GeoGebra before moving to the next question on the sheet as they will use them during the next lessons while exploring the views from top and sides. Figure 9 presents a possible correct construction of the third representation and its front view.

Figure 8. Question – III

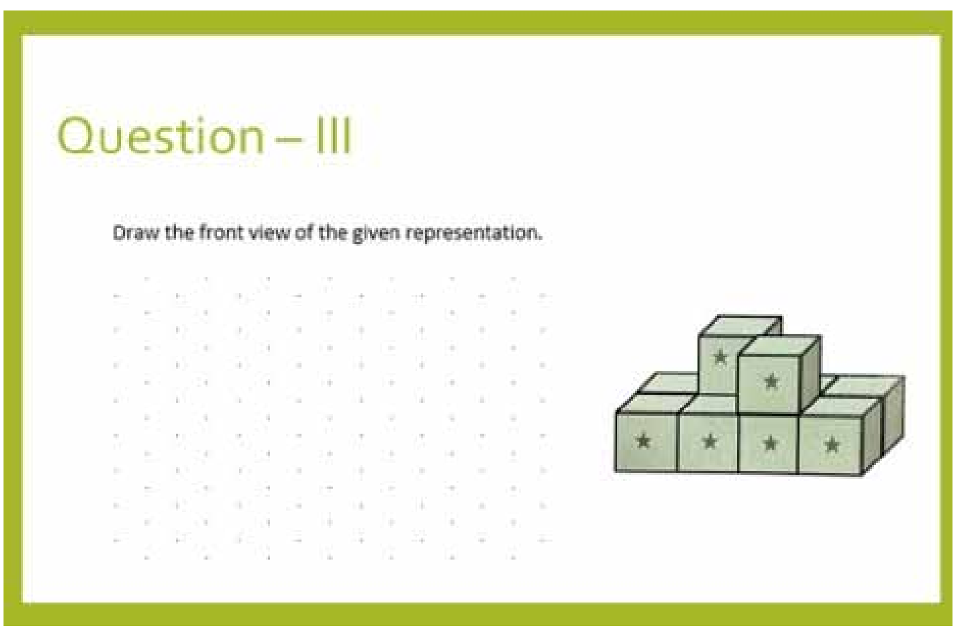
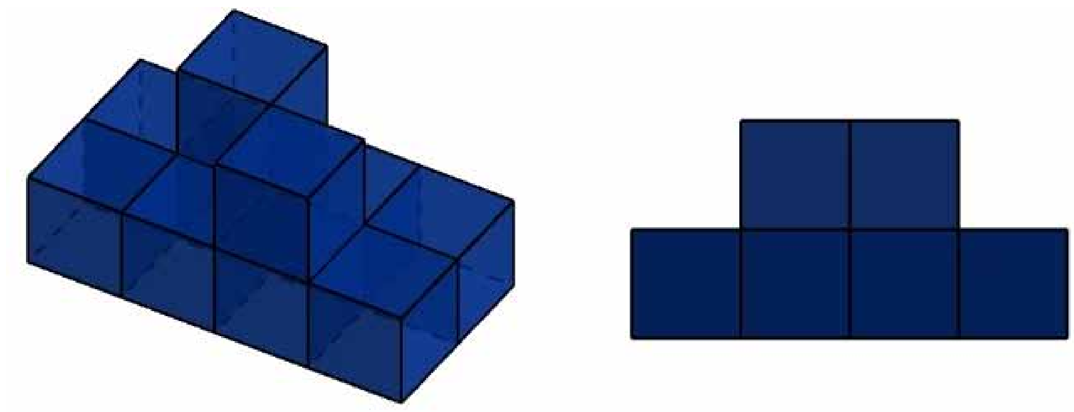
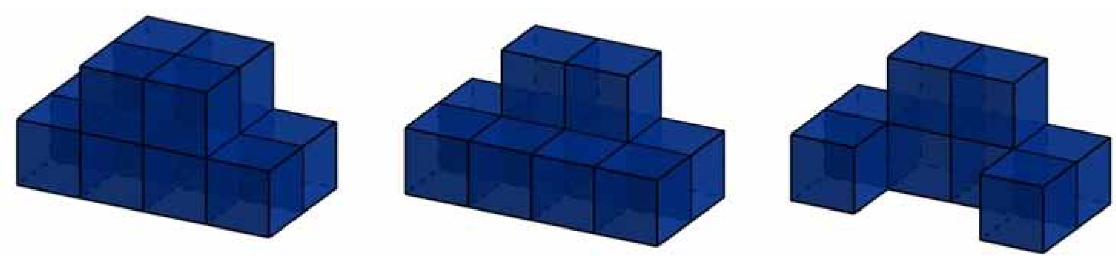


Figure 9. A possible correct GeoGebra construction and its front view – III



Ask students to construct other shapes having the same front view. Say that students can choose to use linking cubes or GeoGebra to construct the shape. Figure 10 presents possible answers.

Figure 10. Possible correct GeoGebra constructions



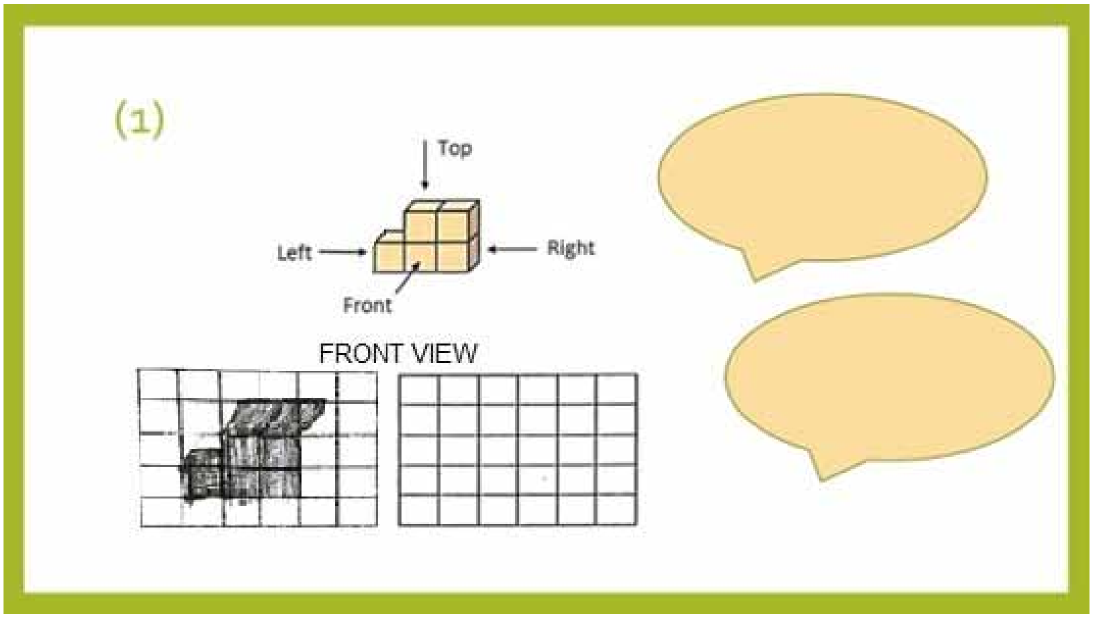
Activity 2

Provide one activity sheet to each student. Activity sheets could have five worked examples, specifically designed with mistakes. Explain to students that now we will explore common student errors in orthogonal and isometric drawings.

Show the slide – Find the mistake and discuss why – I in Figure 11. Say that here is Deniz's work. Ask them the following question. What is the mistake in the drawing and why is that?

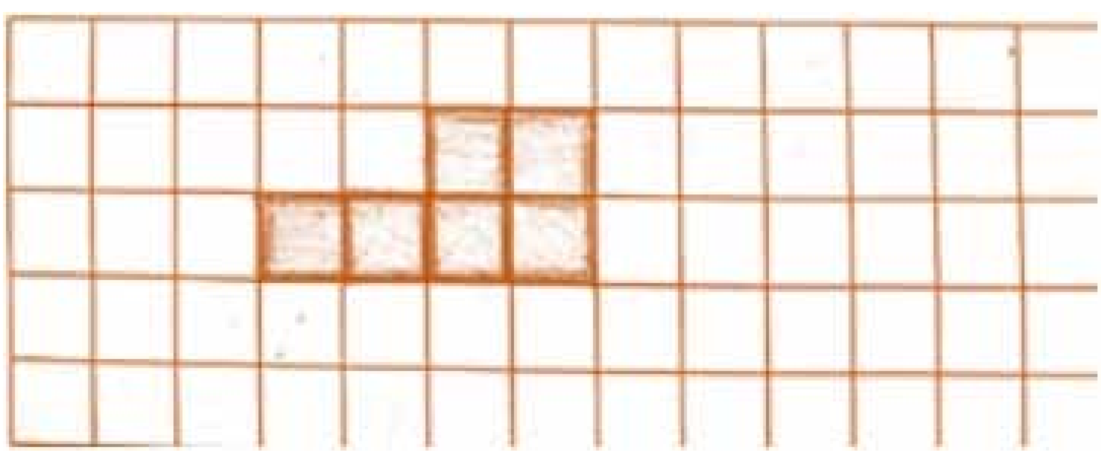
Note that it is better to use "what was he thinking" question instead of asking "what do you think he thought" question to push students to step into the other person's shoes a bit more.

Figure 11. Worked example



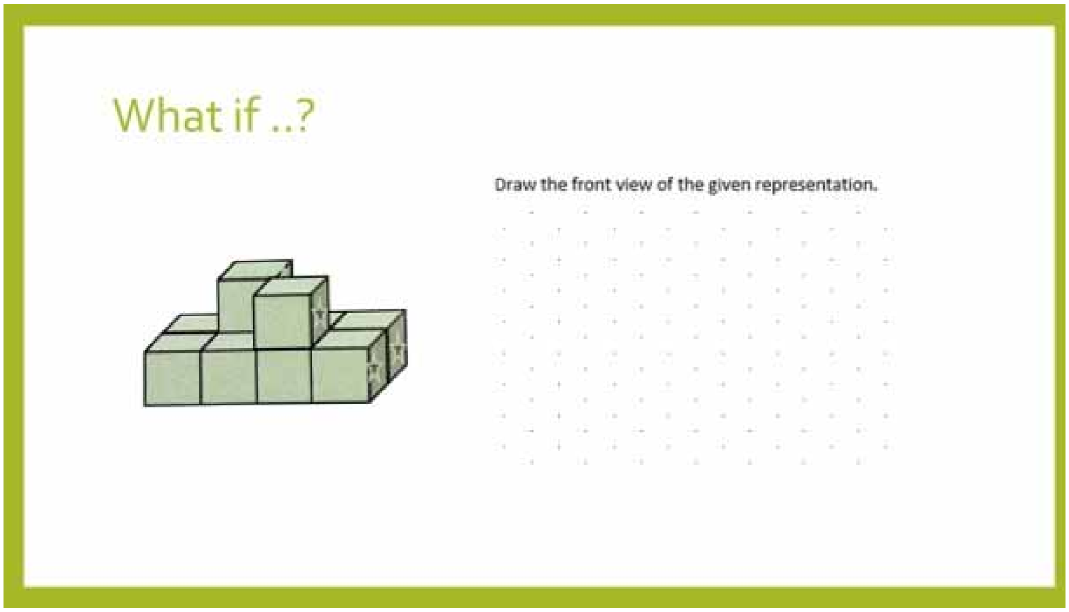
Ask students to discuss why it's wrong in their groups and to note the reason for it to their worksheets. Invite students to share their ideas with the whole class. Ask them to draw the correct representation individually. The correct drawing for the first question is represented in Figure 12. Repeat the procedure for the other four questions.

Figure 12. A possible correct drawing for the worked example - I



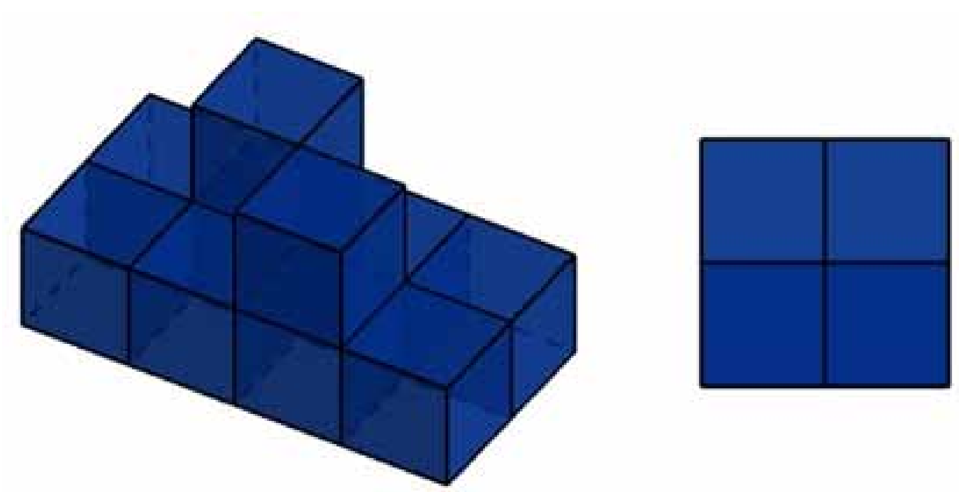
Conclusion and feedback: Conclude the lesson with a question. Show the slide What if in Figure 13, and distribute the sheets of paper to students and ask them to draw the front view of the given shape on the dotted paper.

Figure 13. What if question



This question is different than the other questions and the stars indicating the front view are not in the front perspective. The aim is to raise awareness that front views of polycubical shapes can change according to the perspective we look at. Some students might tend to draw the front view similar to the front view of the third question since the shape actually is the same and the only change is that the stars indicating the front view. Ask them to go back to their GeoGebra constructions and manipulate their constructed shapes to indicate the front view. A possible correct construction of the given representation and its front view are represented in Figure 14.

Figure 14. A possible correct GeoGebra construction and its front view – IV



Explain to students that in this lesson we looked at real-life examples where orthogonal drawings are used, and a real-life example where a dynamic tool was used for drawing. Then, we constructed 3D shapes from linking cubes to represent some buildings in the given pictures mathematically. We explored an authoring tool and represented polycubical shapes on it, and used the tool to draw the view from the front on the dotted paper. The next lesson will be about the views from the top and sides.